

HYPERBOLICITY AND EXPONENTIAL CONVERGENCE OF THE LAX OLEINIK SEMIGROUP FOR TIME PERIODIC LAGRANGIANS

HÉCTOR SÁNCHEZ-MORGADO

ABSTRACT. In this note we prove exponential convergence to time-periodic states of the solutions of time-periodic Hamilton-Jacobi equations on the torus, assuming that the Aubry set is the union of a finite number of hyperbolic periodic orbits of the Euler Lagrange flow. The period of limiting solutions is the least common multiple of the periods of the orbits in the Aubry set. This extends a result that we obtained in [IS] for the autonomous case.

1. INTRODUCTION

Let M be a closed connected manifold, TM its tangent bundle. Let $L : TM \times \mathbb{R} \rightarrow \mathbb{R}$ be a C^k $k \geq 2$ Lagrangian. We will assume for the Lagrangian the hypothesis of Mather's seminal paper [M].

- (1) *Convexity.* The Lagrangian L restricted to $T_x M$, in linear coordinates has positive definite Hessian.
- (2) *Superlinearity.* For some Riemannian metric we have

$$\lim_{|v| \rightarrow \infty} \frac{L(x, v, t)}{|v|} = \infty,$$

uniformly on x and t .

- (3) *Periodicity.* The Lagrangian is periodic in time, i.e.

$$L(x, v, t + 1) = L(x, v, t),$$

for all x, v, t .

- (4) *Completeness.* The Euler Lagrange flow ϕ_t associated to the Lagrangian is complete.

Let $H : T^*M \times \mathbb{R} \rightarrow \mathbb{R}$ be the Hamiltonian associated to the Lagrangian:

$$(1) \quad H(x, p, t) = \max_{v \in T_x M} pv - L(x, v, t).$$

1991 *Mathematics Subject Classification.* 37J50, 49L25, 70H20.

Key words and phrases. viscosity solution. Hamilton-Jacobi equation, periodic Lagrangian, Aubry set.

For $c \in \mathbb{R}$ consider the Hamilton Jacobi equation

$$(2) \quad u_t + H(x, d_x u, t) = c.$$

It is known ([CIS], [B1]) that there is only one value $c = c(L)$, the so called critical value, such that (2) has a time periodic viscosity solution. In this article we prove

Theorem. *Assume $M = \mathbb{T}^d$ and the Aubry set is the union of a finite number of hyperbolic periodic orbits of the Euler Lagrange flow and let N be the least common multiple of their periods. There is $\mu > 0$ such that for any viscosity solution $v : \mathbb{T}^d \times [s, \infty[\rightarrow \mathbb{R}$ of (2) with $c = c(L)$, there is an N -periodic viscosity solution $u : \mathbb{T}^d \times \mathbb{R} \rightarrow \mathbb{R}$ of (2) and $K > 0$ such that for any $t > 0$*

$$|v(x, t) - u(x, t)| \leq K e^{-\mu t}$$

In [IS] we proved a similar result in the autonomous case under the assumption that the Aubry set consists in a finite number of hyperbolic critical points of the Euler Lagrange flow and claimed wrongly, as observed in [WJ], that it also holds when one changes some critical points by periodic orbits. Part of the present note follows same ideas as in the autonomous case, but it was necessary to provide proofs of some statements in that part, which are the contents of Lemma 3 and Proposition 5.

2. PRELIMINARIES ON WEAK KAM THEORY

Define the action of an absolutely continuous curve $\gamma : [a, b] \rightarrow M$ as

$$A_L(\gamma) := \int_a^b L(\gamma(\tau), \dot{\gamma}(\tau), \tau) d\tau$$

A curve $\gamma : [a, b] \rightarrow M$ is *closed* if $\gamma(a) = \gamma(b)$ and $b - a \in \mathbb{Z}$. The critical value can be defined as

$$c(L) := \min\{k \in \mathbb{R} : \forall \gamma \text{ closed } A_{L+k}(\gamma) \geq 0\}$$

Let $\mathcal{P}(L)$ be the set of probabilities on the Borel σ -algebra of $TM \times \mathbb{S}^1$ that have compact support and are invariant under the Euler-Lagrange flow. Then

$$c(L) = -\min\left\{\int L d\mu : \mu \in \mathcal{P}(L)\right\}.$$

The Mather set is defined as

$$\widetilde{M} := \overline{\bigcup\{\text{supp } \mu : \mu \in \mathcal{P}(L), \int L d\mu = -c(L)\}}.$$

For $a \leq b$, $x, y \in M$ let $\mathcal{C}(x, a, y, b)$ be the set of absolutely continuous curves $\gamma : [a, b] \rightarrow M$ with $\gamma(a) = x$ and $\gamma(b) = y$. For $a \leq b$ define $F_{a,b} : M \times M \rightarrow \mathbb{R}$ by

$$F_{a,b}(x, y) := \min\{A_L(\gamma) : \gamma \in \mathcal{C}(x, a, y, b)\}.$$

Define $\mathcal{L}_t : C(M, \mathbb{R}) \rightarrow C(M, \mathbb{R})$ by

$$\mathcal{L}_t u(x) = \inf_{y \in M} u(y) + F_{0,t}(y, x).$$

Then $v(x, t) = \mathcal{L}_t u(x)$ is the viscosity solution to (2) for $c = 0$ with $v(x, 0) = u(x)$. $(\mathcal{L}_n)_{n \in \mathbb{N}}$ is a semi-group known as the Lax-Oleinik semi-group.

For $t \in \mathbb{R}$ let $[t]$ be the corresponding point in \mathbb{S}^1 and $\llbracket t \rrbracket$ be the integer part of t . Define the *action functional* $\Phi : M \times \mathbb{S}^1 \times M \times \mathbb{S}^1 \rightarrow \mathbb{R}$ by

$$\Phi(x, [s], y, [t]) := \min\{F_{a,b}(x, y) + c(L)(b - a) : [a] = [s], [b] = [t]\},$$

and the *Peierls barrier* $h : M \times \mathbb{S}^1 \times M \times \mathbb{S}^1 \rightarrow \mathbb{R}$ by

$$(3) \quad h(x, [s], y, [t]) := \liminf_{\llbracket b-a \rrbracket \rightarrow \infty} (F_{a,b}(x, y) + c(L)(b - a))_{[a]=[s], [b]=[t]}$$

We have $-\infty < \Phi \leq h < \infty$.

The critical value is the unique number c such that (2) has viscosity solutions $u : M \times \mathbb{S}^1 \rightarrow \mathbb{R}$. In fact [CIS], for any $(p, [s])$ the functions $(x, [t]) \mapsto h(p, [s], x, [t])$ and $(x, [t]) \mapsto -h(x, [t], p, [s])$ are respectively *backward* and *forward* viscosity solutions of (2). Set $c = c(L)$ and let $\mathcal{S}^-(\mathcal{S}^+)$ be the set of *backward* (*forward*) viscosity solutions of (2).

The Lagrangian is called regular if the \liminf in (3) is a \lim and in that case, for each $s, t \in \mathbb{R}$ the convergence of the sequence $(F_{a,b})_{[a]=[s], [b]=[t]}$ is uniform. The Lagrangian is regular if and only if the Lax-Oleinik semi-group converges [B1], so that for each $u \in C(M, \mathbb{R})$ there exists $\bar{u} \in C(M \times \mathbb{S}^1, \mathbb{R})$ such that

$$\lim_{k \rightarrow \infty} \mathcal{L}_{t+k} u(x) + c(t+k) = \bar{u}(x, [t])$$

and in fact

$$\bar{u}(x, [t]) = \inf_{y \in M} u(y) + h(y, [0], x, [t]).$$

A subsolution of (2) always means a viscosity subsolution. A curve $\gamma : I \rightarrow M$ *calibrates* a subsolution $u : M \times \mathbb{S}^1 \rightarrow \mathbb{R}$ of (2) if

$$u(\gamma(b), [b]) - u(\gamma(a), [a]) = A_{L+c}(\gamma|[a, b])$$

for any $[a, b] \subset I$. If $u \in \mathcal{S}^-(\mathcal{S}^+)$, for any $(x, [s]) \in M \times \mathbb{S}^1$ there is $\gamma :]-\infty, s] \rightarrow M$ ($\gamma : [s, \infty[\rightarrow M$) that calibrates u and $\gamma(s) = x$.

A pair $(u_-, u_+) \in \mathcal{S}^- \times \mathcal{S}^+$ is called *conjugated* if $u_- = u_+$ on $\mathcal{M} = \pi(\widetilde{\mathcal{M}})$. For such a pair (u_-, u_+) , we define $I(u_-, u_+)$ as the set where u_- and u_+ coincide. If $(x, [s]) \in I(u_-, u_+)$ then u_{\pm} is differentiable at $(x, [s])$ and $d_x u_-(x, [s]) = d_x u_+(x, [s])$. Let

$$I^*(u_-, u_+) = \{(x, d_x u_{\pm}(x, [s]), [s]) : (x, [s]) \in I(u_-, u_+)\}.$$

We may define the *Aubry set* either as the set ([B])

$$\mathcal{A}^* := \bigcap \{I^*(u_-, u_+) : (u_-, u_+) \text{ conjugated}\} \subset T^*M \times \mathbb{S}^1$$

or as its pre-image under the Legendre transformation ([F])

$$\tilde{\mathcal{A}} := \{(x, H_p(x, p, t), [t]) : (x, p, [t]) \in \mathcal{A}^*\}.$$

The projection of either Aubry set in $M \times \mathbb{S}^1$ is

$$\mathcal{A} = \{(x, [t]) \in M \times \mathbb{S}^1 : h(x, [t], x, [t]) = 0\}.$$

An important tool for the proof of our result is the existence of strict C^k critical subsolutions in our setting, that extends the result of Bernard [B] for the autonomous case and its written out in the thesis of E. Guerra [G].

Theorem 1. *Assume the Aubry set $\tilde{\mathcal{A}}$ is the union of a finite number of hyperbolic periodic orbits of the the Euler Lagrange flow, then there is a C^k subsolution f of (2) such that*

$$f_t + H(x, d_x f(x, [t]), t) < c$$

for any $(x, [t]) \notin \mathcal{A}$.

3. REDUCTION TO A REGULAR LAGRANGIAN

Let $M = \mathbb{T}^d$ and assume the Aubry set $\tilde{\mathcal{A}}$ is the union of the hyperbolic periodic orbits

$$\Gamma_i(t) = \phi_t(x_i, v_i, [0]) = (\gamma_i(t), \dot{\gamma}_i(t), [t]) \quad i \in [1, m]$$

of the Euler Lagrange flow with periods $N_i, i \in [1, m]$. In this case the projected Aubry and Mather set coincide. Let N be the least common multiple of N_1, \dots, N_m . Define

$$(4) \quad \begin{aligned} P_N : \mathbb{T}^d \times \mathbb{R}^d \times \mathbb{S}^1 &\rightarrow \mathbb{T}^d \times \mathbb{R}^d \times \mathbb{S}^1 \\ (x, v, [t]) &\mapsto (x, \frac{v}{N}, [Nt]), \end{aligned}$$

and the Lagrangian $L_N = L \circ P_N$. The corresponding Hamiltonian is given by

$$H_N(x, p, t) = H(x, Np, Nt).$$

For a curve $\gamma : [a, b] \rightarrow \mathbb{T}^d$ define $\gamma^N : [a/N, b/N] \rightarrow \mathbb{T}^d, t \mapsto \gamma(Nt)$, then $NA_{L_N}(\gamma^N) = A_L(\gamma)$. A curve γ is an extremal (minimizer) of L if and only if the curve γ^N is an extremal (minimizer) of L_N .

Let $\gamma_{i,j}^N(t) = \gamma_i(j + Nt), j \in [1, N_i], i \in [1, m]$. According to sections 3, 5 of [B1], the Aubry set of L_N is the union of the hyperbolic 1-periodic orbits $\Gamma_{i,j}^N(t) = (\gamma_{i,j}^N(t), \dot{\gamma}_{i,j}^N(t), [t])$ and L_N is regular. Observe that a function $u : \mathbb{T}^d \times \mathbb{R} \rightarrow \mathbb{R}$ is a viscosity solution of (2) if and only if $w(x, t) = \frac{1}{N}u(x, Nt)$ is a viscosity solution of

$$w_t + H_N(x, d_x w, t) = 0.$$

Thus our main Theorem is reduced to the case in which the Lagrangian is regular and the Aubry set is the union of finite number of hyperbolic 1-periodic orbits.

Let $f : \mathbb{T}^{d+1} \rightarrow \mathbb{R}$ be a C^k subsolution of (2) strict outside \mathcal{A} . Consider the Lagrangian

$$\mathbb{L}(x, v, t) = L(x, v, t) - d_x f(x, [t])v - f_t(x, [t]) + c$$

with Hamiltonian $\mathbb{H}(x, p, t) = H(x, p + d_x f(x, [t])) + f_t(x, [t]) - c$.

If $\alpha \in \mathcal{C}(x, s, y, t)$, $A_{\mathbb{L}}(\alpha) = A_{L+c}(\alpha) + f(x, [s]) - f(y, [t])$. Thus L and \mathbb{L} have the same Euler Lagrange flow and projected Aubry set. Moreover,

$$(5) \quad \forall (x, v, t) \quad \mathbb{L}(x, v, t) \geq 0, \quad \tilde{\mathcal{A}} = \{(x, v, t) : \mathbb{L}(x, v, t) = 0\}$$

and u is a viscosity solution of (2) if and only if $u - f$ is a viscosity solution of

$$w_t + \mathbb{H}(x, d_x w, t) = 0.$$

We can therefore assume that $c = 0$, L is regular and has the property (5) of \mathbb{L} , and the Aubry set is the union of hyperbolic 1-periodic orbits $\Gamma_i, i \in [1, m]$ of the Euler Lagrange flow.

Thus, for any $u \in C(\mathbb{T}^d, \mathbb{R})$ we have

$$(6) \quad \lim_{k \rightarrow \infty} \mathcal{L}_{\tau+k} u(x) = \bar{u}(x, [\tau]) := \min_{y \in \mathbb{T}^d} u(y) + h(y, [0], x, [\tau]).$$

We get our result by proving that the convergence in (6) is exponentially fast. The following Lemmas will be helpful

Lemma 2. *Let $W = \bigcup_{i=1}^m W_i$ be a neighborhood of the Aubry set in $\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{S}^1$.*

Then, there exist $C > 0$ such that if $\gamma : [0, t] \rightarrow \mathbb{T}^d$, $t > 1$ is a minimizer, then the time that $(\gamma, \dot{\gamma}, [\cdot])$ remains outside W is less than C .

Proof. Since the Lagrangian is non-negative and it vanishes only on the Aubry set, outside the neighborhood W it is bounded from below by $\eta > 0$. For $t > 1$ the action of minimizers $\gamma : [0, t] \rightarrow \mathbb{T}^d$ is bounded from above independently of t . The Lemma follows. \square

Lemma 3. *Let $V = \bigcup_{i=1}^m V_i$ be a neighborhood of the Aubry set in $\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{S}^1$. There is $N = N(V) \in \mathbb{N}$ such that if $\gamma : [0, t] \rightarrow \mathbb{T}^d$, $t > 1$ is minimizer, then $(\gamma, \dot{\gamma}, [\cdot])$ stays in one V_i during an interval larger than $\frac{t}{N} - 1$.*

Proof. Let $\delta \in]0, 1[$ be such that the δ -neighborhood of Γ_i is contained in V_i for $i \in [1, m]$. Let W_i be the $\frac{\delta}{2}$ -neighborhood of Γ_i and apply Lemma 2 to $W = \bigcup_{i=1}^m W_i$.

Since the velocity of any minimizer $\gamma : [0, t] \rightarrow \mathbb{T}^d$, $t > 1$ is bounded by a constant ≥ 1 , the number of times it can go from W to the complement of V is bounded by some $N \in \mathbb{N}$. The Lemma follows. \square

Let $\lambda_{i,j}, j = 1, \dots, d$ be the positive Lyapunov exponents of γ_i , set $\lambda < \min_{i,j} \lambda_{i,j}$ and write $\phi_t(x, v, [0]) = \psi_t(x, v)$. There is a splitting $\mathbb{R}^{2d} = E_i^- \oplus E_i^+$, invariant under $D_i = d\psi_1(x_i, v_i)$, and a norm $\|\cdot\|$ such that $\|D_i^{\pm 1}|E_i^\mp\| \leq e^{-\lambda}$.

Proposition 4 ([BV], [Be]). *There are $\alpha, \rho \in (0, 1)$, neighborhoods A_i of (x_i, v_i) in $\mathbb{T}^d \times \mathbb{R}^d$, and α -Hölder maps $g_i : A_i \rightarrow B_{2\rho}(0) \subset \mathbb{R}^{2d}$ with α -Hölder inverse such that*

$$D_i \circ g_i = g_i \circ \psi_1.$$

Set

$$(7) \quad U_i = g_i^{-1}(B_\rho), \quad V_i = \bigcup_{s \in [0,1]} (\psi_s(U_i) \cap \psi_{s-1}(U_i)) \times [s], \quad V = \bigcup_{i=1}^m V_i.$$

4. PROOF OF THE RESULT

For brevity, for $x \in \mathbb{T}^d$ we write $\bar{x} = (x, [0])$, and for $\beta : I \rightarrow \mathbb{T}^d$ we write $\bar{\beta}(t) = (\beta(t), [t])$, $d\beta(t) = (\beta(t), \dot{\beta}(t))$, $B(t) = (\beta(t), \dot{\beta}(t), [t])$.

For each $z \in \mathbb{T}^{d+1}$ let y_z be such that

$$\bar{u}(z) = u(y_z) + h(\bar{y}_z, z)$$

In particular let $y_j = y_{\bar{x}_j}$. Since $z \mapsto -h(z, \bar{x}_j) \in \mathcal{S}^+$, there is a semistatic curve $\beta^j : [0, \infty[\rightarrow \mathbb{T}^d$ such that $\beta^j(0) = y_j$ and

$$A_L(\beta^j|[0, t]) = h(\bar{y}_j, \bar{x}_j) - h(\bar{\beta}^j(t), \bar{x}_j), \quad t > 0.$$

We can take $\rho > 0$ given in Proposition 4 such that

$$\{B^j(0) : j \in [1, m]\} \cap V - \tilde{\mathcal{A}} = \emptyset$$

Fix $j \in [1, m]$, set $\beta_0 = \beta^j$ and let Γ_{k_1} be the ω -limit of $B_0 = B^j$. If $k_1 = j$ we stop, otherwise we observe that by the regularity of L

$$h(\bar{\beta}_0(t), \bar{x}_j) = h(\bar{\beta}_0(t), \bar{x}_{k_1}) + h(\bar{x}_{k_1}, \bar{x}_j), \quad t \geq 0$$

and then

$$A_L(\beta_0|[0, t]) = h(\bar{y}_j, \bar{x}_{k_1}) - h(\bar{\beta}_0(t), \bar{x}_{k_1}), \quad t > 0.$$

Proposition 5. *If $k_1 \neq j$, there are $k_1, \dots, k_l = j$ all different and semistatic curves $\beta_r : \mathbb{R} \rightarrow \mathbb{T}^d$, $r < l$ such that Γ_{k_r} and $\Gamma_{k_{r+1}}$ are the α and ω limits of B_r ,*

$$h(\bar{x}_{k_1}, \bar{x}_j) = \sum_{r=1}^{l-1} h(\bar{x}_{k_r}, \bar{x}_{k_{r+1}}),$$

$$A_L(\beta_r|[t, s]) = h(\bar{x}_{k_r}, \bar{\beta}_r(s)) - h(\bar{x}_{k_r}, \bar{\beta}_r(t)), \quad t \leq s.$$

Proof. There is a neighborhood U'_i of $\bar{\gamma}_i$ where $z \mapsto h(\bar{x}_i, z)$ is C^k and the local weak unstable manifold of Γ_i^* is the graph of $d_x h(\bar{x}_i, x, [t])$. Let U_i be a neighborhood of $\bar{\gamma}_i$ with compact $\bar{U}_i \subset U'_i$. Let $\rho_n : [0, n] \rightarrow \mathbb{T}^d$ be a curve joining x_{k_1} to x_j such that

$$A_L(\rho_n) = F_{0,n}(x_i, x_j).$$

Let $t_n \in [0, n]$ be the first exit time of $\bar{\gamma}_n(t)$ out of U_i , and $\bar{\gamma}_n(t_n)$ be the first point of intersection with ∂U_{k_1} . As n goes to infinity, t_n and $n - t_n$ tend to infinity. This follows from the fact that $\dot{\rho}_n(0)$ has to tend to $\dot{\gamma}_{k_1}(0)$, and $\dot{\rho}_n(n)$ has to tend to $\dot{\gamma}_j(0)$. To justify this, consider v a limit point of $\dot{\rho}_n(0)$, and $\gamma : \mathbb{R} \rightarrow \mathbb{T}^d$ the solution to the Euler-Lagrange equation such that $\gamma(0) = x_i, \dot{\gamma}(t) = v$. From the fact that

$$F_{0,n}(x_{k_1}, x_j) - F_{1,n}(\rho_n(1), x_j) = A_L(\rho_n|_{[0,1]})$$

and the regularity of L , taking limit $n \rightarrow \infty$ it follows

$$h(\bar{x}_{k_1}, \bar{x}_j) - h(\bar{\gamma}(1), \bar{x}_j) = A_L(\gamma|_{[0,1]}).$$

Since $\gamma_{k_1}(-1) = x_{k_1}$ and $L = 0$ on $\tilde{\mathcal{A}}$

$$h(\bar{\gamma}_{k_1}(-1), \bar{x}_j) - h(\bar{\gamma}(1), \bar{x}_j) = A_L(\gamma_{k_1}|_{[-1,0]}) + A_L(\gamma|_{[0,1]})$$

so that the curve obtained by gluing $\gamma_{k_1}|_{[-1,0]}$ with $\gamma|_{[0,1]}$ minimizes the action between its endpoints. In particular, it has to be differentiable, thus $v = \dot{\gamma}(0) = \dot{\gamma}_{k_1}(0)$. Define $\alpha_n : [-\llbracket t_n \rrbracket, n - \llbracket t_n \rrbracket] \rightarrow \mathbb{T}^d$ by $\alpha_n(t) = \rho_n(t + \llbracket t_n \rrbracket)$ and let (y, w, τ) be a cluster point of $(d\rho_n(t_n), t_n - \llbracket t_n \rrbracket)$. Then there is a sequence (α_{n_i}) converging uniformly on compact intervals to the solution $\beta_1 : \mathbb{R} \rightarrow \mathbb{T}^d$ of the Euler-Lagrange equation such that $d\beta_1(\tau) = (y, w)$. Since for any $t \leq s$ we have

$$\begin{aligned} F_{-\llbracket t_n \rrbracket, s}(x_{k_1}, \alpha_n(s)) - F_{-\llbracket t_n \rrbracket, t}(\bar{x}_{k_1}, \alpha_n(t)) &= A_L(\alpha_n|_{[t, s]}), \\ F_{-\llbracket t_n \rrbracket, t}(x_{k_1}, \alpha_n(t)) + F_{t, n - \llbracket t_n \rrbracket}(\alpha_n(t), x_j) &= F_{0,n}(x_{k_1}, x_j), \end{aligned}$$

from the uniform convergence of $F_{a,b}$ when $\llbracket b - a \rrbracket \rightarrow \infty$, we obtain for any $t \leq s$

$$\begin{aligned} h(\bar{x}_{k_1}, \bar{\beta}_1(s)) - h(\bar{x}_{k_1}, \bar{\beta}_1(t)) &= A_L(\beta_1|_{[t, s]}) \\ h(\bar{x}_{k_1}, \bar{\beta}_1(t)) + h(\bar{\beta}_1(t), \bar{x}_j) &= h(\bar{x}_{k_1}, \bar{x}_j). \end{aligned}$$

Since $\bar{\alpha}_n([-\llbracket t_n \rrbracket, t_n - \llbracket t_n \rrbracket]) \subset \bar{U}_{k_1}$ we have that Γ_{k_1} is the α -limit of B_1 and let Γ_{k_2} be its ω -limit. If $k_2 = j$ we stop, otherwise we observe that

$$\begin{aligned} h(\bar{x}_{k_1}, \bar{x}_{k_2}) + h(\bar{x}_{k_2}, \bar{x}_j) &= h(\bar{x}_{k_1}, \bar{x}_j), \\ h(\bar{x}_{k_1}, \bar{\beta}_1(t)) + h(\bar{\beta}_1(t), \bar{x}_{k_2}) &= h(\bar{x}_{k_1}, \bar{x}_{k_2}). \end{aligned}$$

Therefore $k_1 \neq k_2$. We proceed in the same way to find a solution to the Euler-Lagrange equation $\beta_2 : \mathbb{R} \rightarrow \mathbb{T}^d$ such that Γ_{k_2} is the α -limit of B_2 and for any $t, s \in \mathbb{R}, t < s$

$$\begin{aligned} h(\bar{x}_{k_2}, \bar{\beta}_2(s)) - h(\bar{x}_{k_2}, \bar{\beta}_2(t)) &= A_L(\beta_2|_{[t, s]}) \\ h(\bar{x}_{k_2}, \bar{\beta}_2(t)) + h(\bar{\beta}_2(t), \bar{x}_j) &= h(\bar{x}_{k_2}, \bar{x}_j). \end{aligned}$$

Let Γ_{k_3} be the ω -limit of B_2 . If $k_3 = j$ we stop, otherwise we observe that

$$\begin{aligned} h(\bar{x}_{k_2}, \bar{x}_{k_3}) + h(\bar{x}_{k_3}, \bar{x}_j) &= h(\bar{x}_{k_2}, \bar{x}_j), \\ h(\bar{x}_{k_2}, \bar{\beta}_2(t)) + h(\bar{\beta}_2(t), \bar{x}_{k_3}) &= h(\bar{x}_{k_2}, \bar{x}_{k_3}). \end{aligned}$$

Therefore $k_2 \neq k_3$. Since

$$h(\bar{x}_{k_1}, \bar{x}_{k_2}) + h(\bar{x}_{k_2}, \bar{x}_{k_3}) = h(\bar{x}_{k_1}, \bar{x}_{k_3}),$$

we have that $k_1 \neq k_3$. We continue until we get $k_r = j$. \square

Let V be given by (7). There is $T > 0$ such that for $t \geq T$, we have $B_r(t), B_r(-t) \in V$ and then

$$\begin{aligned} d(d\beta_r(t), d\gamma_{k_{r+1}}(t)) &\leq C_1 e^{-\lambda t}, \quad t > T \\ d(d\beta_r(t), d\gamma_{k_r}(t)) &\leq C_1 e^{\lambda t}, \quad t < -T. \end{aligned}$$

For $(x, \tau) \in \mathbb{T}^d \times [0, 1]$, we consider a directed graph with vertices at the points $\bar{x}_1, \dots, \bar{x}_m$ in the Aubry set, and a directed segment from \bar{x}_j to \bar{x}_k if and only if

$$h(\bar{x}_k, x, [\tau]) = h(\bar{x}_k, \bar{x}_j) + h(\bar{x}_j, x, [\tau]).$$

We call a point \bar{x}_k a root of this graph if there is no segment arriving to this point and this means that for $j \neq k$

$$h(\bar{x}_k, x, [\tau]) < h(\bar{x}_k, \bar{x}_j) + h(\bar{x}_j, x, [\tau]).$$

Notice that the graph contains no cycles, and so each point \bar{x}_k belongs to a branch starting at a root. If $\bar{u}(x, [\tau]) = \bar{u}(\bar{x}_k) + h(\bar{x}_k, x, [\tau])$ and there is a segment from x_j to x_k , then

$$\begin{aligned} \bar{u}(x, [\tau]) &\leq \bar{u}(\bar{x}_j) + h(\bar{x}_j, x, [\tau]) \leq \bar{u}(\bar{x}_k) + h(\bar{x}_k, \bar{x}_j) + h(\bar{x}_j, x, [\tau]) \\ &= \bar{u}(\bar{x}_k) + h(\bar{x}_k, x, [\tau]) = \bar{u}(x, [\tau]). \end{aligned}$$

Take $k \in [1, m]$ such that

$$\bar{u}(x, [\tau]) = \bar{u}(\bar{x}_k) + h(\bar{x}_k, x, [\tau])$$

and let \bar{x}_j be the root of a branch containing \bar{x}_k . Since $z \mapsto h(\bar{x}_j, z) \in \mathcal{S}^-$, there is a semistatic curve $\beta_{x,\tau} :]-\infty, \tau] \rightarrow \mathbb{T}^d$ with $\beta_{x,\tau}(\tau) = x$ such that

$$A_L(\beta_{x,\tau}|[t, \tau]) = h(\bar{x}_j, x, [\tau]) - h(\bar{x}_j, \bar{\beta}_{x,\tau}(t)), \quad t < \tau.$$

Let Γ_i be the α -limit of $B_{x,\tau}$. If $(x, [\tau]) = \bar{\gamma}_j(\tau)$ then $\beta_{x,\tau} = \gamma_i = \gamma_j$. By the regularity of L

$$h(\bar{x}_j, x, [\tau]) = h(\bar{x}_j, \bar{x}_i) + h(\bar{x}_i, x, [\tau]),$$

since \bar{x}_j is a root, $i = j$.

From Lemma 2 there is $\bar{T} \geq T$ such that $B_{x,\tau}(t) \in V_j$ for $t \leq -\bar{T}$

$$d(d\beta_{x,\tau}(t), d\gamma_j(t)) \leq C_1 e^{\lambda t},$$

unless $B_{x,\tau}$ is part of an orbit of the Euler-Lagrange flow with ω -limit some Γ_i and $B_{x,\tau}(\tau) \in V_i$. If the last situation occurs let $M = \max\{q \in \mathbb{N} : B_{x,\tau}([-q, \tau]) \subset V_i\}$ so that $d(d\beta_{x,\tau}(t), d\gamma_i(t)) \leq C_2 e^{-\lambda(M+t)}$. Otherwise take $M = 0$.

For $k \in \mathbb{N}$ let $n + 1 = \left\lceil \frac{k}{2(l+1)} \right\rceil$ and define the curve $\alpha_k : [0, \tau + k] \rightarrow \mathbb{T}^d$ by

$$\alpha_k(s) = \begin{cases} \beta_0(s) & s \in [0, n] \\ c_r(s - r(2n+1) + n + 1) & s \in [r(2n+1) - n - 1, r(2n+1) - n], r \leq l \\ \beta_r(s - r(2n+1)) & s \in [r(2n+1) - n, (r+1)(2n+1) - n - 1] \\ \gamma_j(s) & s \in [l(2n+1) - n, k - 2n - 1] \\ c_{l+1}(s - k + 2n + 1) & s \in [k - 2n - 1, k - 2n] \\ \beta_{x,\tau}(s - k - (M - n)_+) & s \in [k - 2n, \tau + k + (M - n)_+] \end{cases}$$

where $c_r : [0, 1] \rightarrow \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ is defined for n large by

$$c_r(s) = \begin{cases} (1-s)\beta_{r-1}(s+n) + s\beta_r(s-1-n) & r < l \\ (1-s)\beta_l(s+n) + s\gamma_j(s) & r = l \\ (1-s)\gamma_j(s) + s\beta_{x,\tau}(s-1-n-M) & r = l+1. \end{cases}$$

Notice that if $(x, \tau) = \gamma_j(\tau)$ then $\alpha_k(s) = \gamma_j(s)$ for $s \geq l(2n+1) - n$.

Since $L \geq 0$, when $n \geq M$ we have

$$\begin{aligned} A_L(\alpha_k) &= A_L(\beta_0|_{[0,n]}) + \sum_{r=1}^{l-1} A_L(\beta_r|_{[-n,n]}) + A_L(\beta_{x,\tau}|_{[-2n,\tau]}) + \sum_{r=1}^{l+1} A_L(c_r) \\ &\leq h(\bar{y}_j, \bar{x}_{k_1}) + \sum_{r=1}^{l-1} h(\bar{x}_{k_r}, \bar{x}_{k_{r+1}}) + h(\bar{x}_j, (x, [\tau])) + \sum_{r=1}^{l+1} A_L(c_r) \\ &= h(\bar{y}_j, \bar{x}_j) + h(\bar{x}_j, (x, [\tau])) + \sum_{r=1}^{l+1} A_L(c_r) \end{aligned}$$

Thus

$$\mathcal{L}_{\tau+k}u(x) - \bar{u}(x, [\tau]) \leq \sum_{r=1}^{l+1} A_L(c_r)$$

Since $L = 0$ on $\tilde{\mathcal{A}}$ and there is $C_3 \geq C_2$ with

$$(8) \quad d(dc_r(s), d\gamma_{k_r}(s)), d(dc_{l+1}(s), d\gamma_j(s)) \leq C_3 e^{-\lambda n}$$

we have

$$\mathcal{L}_{\tau+k}u(x) - \bar{u}(x, [\tau]) \leq C_4 e^{-\lambda n} \leq C_5 e^{-\lambda k/2m}.$$

When $M > n$ define

$$\begin{aligned} c_{l+2}(s) &= -s\beta_{x,\tau}(s + \tau + n - M) + (1 + s)\beta_{x,\tau}(s + \tau) \quad s \in [-1, 0] \\ \hat{\alpha}_k(s) &= c_{l+2}(s - k - \tau) \quad s \in [k + \tau - 1, k + \tau]. \end{aligned}$$

Since $L \geq 0$ we have as before

$$A_L(\alpha_k|[0, k + \tau - 1]) + A_L(\hat{\alpha}_k) \leq h(\bar{y}_j, \bar{x}_j) + h(\bar{x}_j, (x, [\tau])) + \sum_{r=1}^{l+2} A_L(c_r)$$

Since $L = 0$ on $\tilde{\mathcal{A}}$, (8) and $d(dc_{l+2}(s), d\gamma_i(s)) \leq C_3 e^{-\lambda n}$ we have

$$\mathcal{L}_{\tau+k}u(x) - \bar{u}(x, [\tau]) \leq C_4 e^{-\lambda n} \leq C_5 e^{-\lambda k/2m}.$$

Remark 6. The points \bar{y}_i depend on the function u . To get a constant C_4 independent of u we should manage situations similar to the case $M > 0$ above for a continuum of points.

We now establish the opposite inequality.

For $x \in M$, $t > 0$ let $\gamma = \gamma_{x,t} : [0, t] \rightarrow \mathbb{T}^d$ be such that $\gamma(t) = x$ and

$$\mathcal{L}_t u(x) = u(\gamma(0)) + A_L(\gamma_t) = u(\gamma(0)) + F_{0,t}(\bar{\gamma}(0), x, [t]).$$

For any integer $j \in [0, t]$, $i \in [1, m]$ we have

$$\begin{aligned} \bar{u}(x, [t]) &\leq \bar{u}(\bar{x}_i) + h(\bar{x}_i, x, [t]) \\ &\leq u(\gamma(0)) + h(\bar{\gamma}(0), x_i) + h(\bar{x}_i, x, [t]) \\ &\leq u(\gamma(0)) + \Phi(\bar{\gamma}(0), \bar{\gamma}(j)) + h(\bar{\gamma}(j), \bar{x}_i) + h(\bar{x}_i, \bar{\gamma}(j)) + \Phi(\bar{\gamma}(j), x, [t]) \\ &\leq u(\gamma(0)) + A_L(\gamma) + h(\bar{\gamma}(j), \bar{x}_i) + h(\bar{x}_i, \bar{\gamma}(j)) \\ &= \mathcal{L}_t u(x) + h(\bar{\gamma}(j), \bar{x}_i) + h(\bar{x}_i, \bar{\gamma}(j)) \\ &\leq \mathcal{L}_t u(x) + Kd(\gamma(j), x_i). \end{aligned}$$

Apply Lemma 3 to V given by (7). If $\gamma : [0, t] \rightarrow \mathbb{T}^d$ is a minimizer and $n + 1 = \left\lfloor \frac{t}{2N} \right\rfloor$, $N = N(V)$, then $(\gamma, \dot{\gamma}, [\cdot])$ stays in one V_i on an interval I of length $2n + 1$. Let $j \in \mathbb{N}$ be such that $[j, j + 2n] \subset I$. Letting $Y_l = (\gamma(j + l), \dot{\gamma}(j + l))$ and writting

$$g_i(Y_0) = z_- + z_+, \quad g_i(Y_{2n}) = w_- + w_+, \quad z_{\pm}, w_{\pm} \in E_i^{\pm}$$

we have

$$\begin{aligned} g_i(Y_n) &= D_i^n(z_-) + D_i^{-n}(w_+), \\ \|g_i(Y_n)\| &\leq e^{-\lambda n}(\|z_-\| + \|w_+\|), \\ d(\gamma(j + n), x_i) &\leq d(Y_n, (x_i, v_i)) \leq C_5 e^{-\alpha \lambda n} \leq C_6 e^{-\alpha \lambda t/2N}. \end{aligned}$$

REFERENCES

- [BV] Barreira, L; Valls, Hölder Grobman-Hartman linearization. *Disc. Cont. Dynam. Sys.* **18** No. 1 (2007) 187-197
- [Be] Belitskii, G. R; On the Grobman-Hartman theorem in class C^α . Unpublished preprint.
- [B] Bernard, P. Smooth critical subsolutions of the Hamilton-Jacobi equation. *Math. Res. Lett.* **14** no. 3, (2007) 503–511.
- [B1] Bernard P. Connecting orbits of time dependent Lagrangian systems. *Ann. Inst. Fourier, Grenoble* **52** no. 5, (2002) 1533-1568
- [CIS] Contreras, G.; Iturriaga, R. Sánchez-Morgado H. Weak solutions of the Hamilton Jacobi equation for Time Periodic Lagrangians. Unpublished preprint.
- [IS] Iturriaga, R. Sánchez-Morgado H. Hiperbolicity and exponential convergence of the Lax Oleinik semigroup. *J. Diff. Eq.* Volume 246 (2009) 17441753. MR 2010j:37087.
- [F] Fathi, A. *The Weak KAM Theorem in Lagrangian Dynamics*. Cambridge Studies in Advanced Mathematics, 2010.
- [G] Guerra, E. Tesis Doctoral UNAM, in preparation.
- [M] Mather, J. Action minimizing measures for positive definite Lagrangian systems. *Math. Z.* **207** (1991) 169–207.
- [WJ] K. Wang, Jun Yan. The rate of convergence of new kind of Lax-Oleinik type operator for time-periodic positive definite Lagrangian systems. ArXiv preprint DS/1109.3327v1.

INSTITUTO DE MATEMÁTICAS, UNAM. CIUDAD UNIVERSITARIA C. P. 04510, CD. DE MÉXICO, MÉXICO.

E-mail address: `hector@math.unam.mx`